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# Superconformal Chern-Simons theories

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Abstract: We explore the possibilities for constructing lagrangian descriptions of three-dimensional superconformal classical gauge theories that contain a Chern-Simons term, but no kinetic term, for the gauge fields. Classes of such theories with $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetry are found. However, interacting theories of this type with $\mathcal{N} = 8$ supersymmetry do not exist.

Keywords: Chern-Simons Theories, AdS-CFT and dS-CFT Correspondence, Supersymmetry and Duality.



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## Contents



## 1. Introduction

Many examples of conformal field theories are known in two dimensions and in four dimensions. However, much less is known in three dimensions. From the perspective of AdS/CFT one is particularly interested in conformally invariant gauge theories, where the rank of the gauge group is related to the amount of flux in the dual AdS description. M theory admits compactifications involving $AdS_4$, the most symmetrical choice being $AdS_4 \times S^7$ [1]. According to the AdS/CFT conjecture [2] this should be dual to a three-dimensional gauge theory with the superconformal symmetry OSp(8|4). This gauge theory should have gauge group U($N$) if the dual M theory background has $N$ units of flux through the seven-sphere. The situation ought to be rather analogous to the case of type IIB superstring theory compactified on $AdS_5 \times S^5$, with $N$ units of flux, for which the dual gauge theory is $\mathcal{N}=4$ supersymmetric Yang-Mills theory with a U($N$) gauge group and the superconformal symmetry is PSU(4|2, 2). There are some significant differences, however. For one thing the type IIB superstring background contains a constant dilaton field, whose value corresponds to the Yang-Mills coupling constant. There is no analogous scalar field in the M theory case. Therefore the dual three-dimensional CFT should not have an adjustable coupling, and therefore it is expected to be strongly coupled. This makes it a logical possibility that there is no explicit lagrangian description of this theory, but it does not imply that this must be the case.

The usual viewpoint, which surely is correct, is the following: The low-energy effective world-volume theory on a collection of $N$ coincident D2-branes of type IIA superstring theory is a maximally supersymmetric U($N$) Yang-Mills theory in three dimensions. This theory, which is not conformal because the Yang-Mills coupling in three dimensions is dimensionful, has an SO(7) R symmetry corresponding to rotations of the transverse directions. In the flow to the infrared the gauge coupling increases, which corresponds to the string coupling (the vev of the dilaton) increasing. This in turn corresponds to the radius of the circular 11th dimension increasing. In the limit that the coupling becomes infinite, one reaches the conformally-invariant fixed point theory that describes a collection of coincident M2-branes in eleven dimensions. This theory should have an enhanced SO(8) R symmetry corresponding to rotations of the eight transverse dimensions. One question that we wish to explore in this paper is whether it is possible to find an alternative characterization of this fixed-point theory with an explicit classical lagrangian.

One can anticipate the field content of these theories from the relation to M2-branes (in the M theory case) and D3-branes (in the type IIB superstring theory case). The world-volume field content of a single D3-brane contains a vector, six scalars, and four Majorana spinors. To describe $N$ coincident D3-branes (at low energy) it is just a matter of promoting these to $N \times N$ hermitean matrices and constructing an interacting superconformal field theory with U($N$) gauge symmetry. This is achieved by $\mathcal{N} = 4$ SYM theory, of course. In the case of an M2-brane, the physical world-volume field content consists of eight scalars and eight (two-component) Majorana spinors. So a natural guess is that these should be made into $N \times N$ matrices and the U($N$) global symmetry should be gauged. However, this is not entirely obvious, because unlike the case of D-branes, there is no simple interpretation in terms of strings stretched between the branes. When viewed in terms of the maximally supersymmetric SYM theory that flows to the desired fixed-point theory one sees this field content except that one of the matrix scalars is replaced by a propagating gauge field. In the abelian case these can be related by a duality transformation, but in the nonabelian case there is no simple way of doing that. Rather than trying to carry out such a duality transformation, we will start with the postulated field content, which is clearly required for exhibiting the desired Spin(8) R symmetry.

The U($N$) gauge theory should have $\mathcal{N} = 8$ super-Poincaré symmetry and scale invariance, which together ought to imply the full OSp(8|4) superconformal symmetry.[1] If one succeeds in constructing such a theory, then it would be reasonable to expect that quantum corrections do not destroy the scale invariance, like in the case of $\mathcal{N} = 4$ SYM theory.

The scalars and spinors in the proposed three-dimensional CFT give an equal number of physical bosonic and fermionic degrees of freedom. Therefore, to maintain supersymmetry when U($N$) gauge fields are added, the number of bosonic degrees of freedom should not change. This should be contrasted with the case of $\mathcal{N} = 4$ SYM theory, where the transverse polarizations of the gauge fields are required to achieve an equal number of bosonic and fermionic degrees of freedom. Starting from the free theory with global U($N$)

[1] Usually, but not always, Poincaré invariance together with scale invariance implies conformal invariance. (See [3] and references therein.)



JHEP11(2004)078

symmetry in three dimensions there are three alternative ways to introduce the gauge fields that one might consider: (1) Add gauge field couplings to make the global $\mathrm{U}(N)$ symmetry local, but do not introduce kinetic terms for the gauge fields. (2) Add gauge field couplings to make the global $\mathrm{U}(N)$ symmetry local and add $F^2$ kinetic terms for the gauge fields. (3) Add gauge field couplings to make the global $\mathrm{U}(N)$ symmetry local and add a Chern-Simons term for the gauge fields. We claim that choice number (1) is inconsistent with supersymmetry, because the gauge fields would give rise to constraints that would effectively subtract bosonic degrees of freedom. Similarly, choice number (2) is unacceptable, because the gauge fields would add bosonic degrees of freedom. Also $F^2$ is dimension 4, and scale invariance of the classical theory only allows dimension 3 terms. This leaves choice (3), which I claim is exactly right. The Chern-Simons term is dimension 3 and its inclusion does not lead to either an increase or a decrease in the number of propagating bosonic degrees of freedom, so it is conceivable that supersymmetry can be achieved.

To be honest, it is quite mysterious how a Chern-Simons term could be generated in the IR flow of the SYM theory discussed earlier. This is especially a concern since the SYM theory that flows to the fixed point in question is parity conserving. So how could the theory be parity violating at the fixed point? In the end, we will not find such an $\mathcal{N} = 8$ theory, and maybe this is one of the reasons why.

As we have said, the problem that we would most like to solve is the explicit construction of a lagrangian for the three-dimensional CFT that has maximal supersymmetry and is dual to M theory on $AdS_4 \times S^7$. However, most of this paper will address more modest goals: the construction of three-dimensional gauge theories with $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetry and classical scale invariance. This will provide a framework for explaining why an $\mathcal{N} = 8$ super Chern-Simons theory cannot be constructed. However, it is conceivable that one could construct a lagrangian description of the desired $\mathrm{OSp}(8|4)$ superconformal theory by modifying one or more of our assumptions.

## 2. Supersymmetry of Chern-Simons theories

Pure Chern-Simons theory has a lagrangian that is proportional to

$$\mathcal{L}_{\mathrm{CS}} = \mathrm{tr}\left[\epsilon^{\mu\nu\rho}\left(A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho\right)\right]. \tag{2.1}$$

It gives the classical field equation

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] = 0\,. \tag{2.2}$$

A curious fact about this theory is that it has any desired amount of supersymmetry, if one simply decrees that $A_\mu$ is invariant under each of the supersymmetry transformations. The reason this is possible is that this theory has no propagating on-shell degrees of freedom. To prove this assertion one needs to verify the super-Poincaré algebra, especially that the commutator of two supersymmetry transformations is a translation. Since the supersymmetry transformation is trivial, this means that the translation symmetry transformation should also be trivial.

Since $A_\mu$ by itself is certainly not a complete off-shell supermultiplet, the supersymmetry algebra should only hold on-shell. This means that in verifying the closure of the algebra, one is allowed to use the field equation $F_{\mu\nu} = 0$. This is a familiar situation; many of the nicest supersymmetric theories, such as $\mathcal{N} = 4$ super Yang-Mills theory, do not have a straightforward formulation in terms of off-shell supermultiplets of the full supersymmetry algebra. So the proof of the assertion that we are making is simply to show that an infinitesimal translation by a constant amount $a^\rho$ is trivial modulo a gauge transformation and the equations of motion. This is the case because an infinitesimal translation shifts $A_\mu$ by $a^\rho \partial_\rho A_\mu$, which differs from $a^\rho F_{\rho\mu}$ by an infinitesimal gauge transformation

$$\delta A_\mu = \nabla_\mu \Lambda = \partial_\mu \Lambda + i[A_\mu, \Lambda] \tag{2.3}$$

for the choice $\Lambda = a^\rho A_\rho$. This then vanishes by the equations of motion. Of course, this triviality of translation invariance is not a big surprise since Chern-Simons is a topological theory.

We will be interested in coupling the Chern-Simons gauge field to other fields. For this purpose it is convenient to have complete off-shell supermultiplets. This enables one to combine supersymmetric expressions without substantial modification of the supersymmetry transformation formulas, as we will see. Pure Chern-Simons theories with off-shell supersymmetry were constructed in [5] for $\mathcal{N} = 1, 2, 4$. That work did not discuss coupling these supermultiplets to other matter supermultiplets.

## 3. $\mathcal{N} = 1$ models

### 3.1 The gauge multiplet

One of the nice things about the $\mathcal{N} = 1$ theories in three dimensions that we want to construct is that it is easy to implement supersymmetry by using superfields. The Grassmann coordinates of $\mathcal{N} = 1$ superspace consist just of a two-component Majorana spinor. There are two kinds of multiplets that we will be interested in: gauge multiplets and scalar multiplets. In this section we discuss the gauge multiplet. This superfield is a spinor. However, in this case we find it convenient to work with the component fields that survive in the three-dimensional analog of Wess–Zumino gauge. These are the gauge field $A_\mu$ and a Majorana two-component spinor $\chi$. Both of these are in the adjoint representation of the Lie algebra and can be represented as hermitean matrices in some convenient representation, which will be specified later when they are coupled to scalar supermultiplets.

Since we are mainly interested in classical considerations in this paper, we will not specify the overall normalization of the action at this time. This would need to be considered carefully in defining the quantum theory, of course. With this understanding we choose the $\mathcal{N} = 1$ Chern-Simons lagrangian to be

$$\mathcal{L}_{\text{CS}} = \text{tr}\left[\epsilon^{\mu\nu\rho}\left(A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho\right) - \bar{\chi}\chi\right]. \tag{3.1}$$

This theory differs from the pure Chern-Simons theory discussed in the preceding section only by the addition of the auxiliary fermi field $\chi$. Note that the lagrangian has dimension three for the choices dim $A = 1$ and dim $\chi = 3/2$, and then the action is scale invariant.

To get off-shell closure of the supersymmetry algebra, one needs to have an equal number of off-shell bosonic and fermionic degrees of freedom. In fact, taking account of gauge invariance, $A_\mu$ and $\chi$ both have two off-shell modes. This ensures off-shell closure of the supersymmetry algebra without use of equations of motion.

The infinitesimal supersymmetry transformations that leave $\mathcal{L}_{\rm CS}$ invariant (up to a total derivative) are[2]

$$\delta A_\mu = i\bar{\varepsilon}\gamma_\mu\chi \tag{3.2}$$

and

$$\delta\chi = \frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\,\varepsilon\,. \tag{3.3}$$

The commutator of two supersymmetry transformations $[\delta_1,\delta_2]$ gives the sum of a spacetime translation by $a^\rho = 2i\bar{\varepsilon}_1\gamma^\rho\varepsilon_2$ and a gauge transformation by $\Lambda = -a^\rho A_\rho$.

### 3.2 The matter theory

Let us now turn to the scalar supermultiplets, which we write in terms of superfields as follows:

$$\Phi = \phi + \bar{\theta}\psi + \frac{1}{2}\bar{\theta}\theta C\,. \tag{3.4}$$

Let us take $\Phi^a$, $a = 1,2,\ldots,\dim R$, to belong to a representation $R$ of the gauge group $G$. We may assume without loss of generality that $R$ is real. Then there is no need to make a distinction between upper and lower indices. In this section we will formulate a scale invariant theory of the scalar superfields with global $G$ symmetry. In the next section we will couple this theory to the gauge supermultiplet, so as to achieve local $G$ symmetry while retaining global $\mathcal{N}=1$ supersymmetry.

To achieve scale invariance we assign dimension 1/2 to $\Phi$. This implies that dim $\phi = 1/2$, dim $\psi = 1$, and dim $C = 3/2$. Then the most general scale-invariant theory is given by the $\bar{\theta}\theta$ component of a dimension two superfield expression. The only possibilities are a kinetic term of the form $\bar{D}\Phi^a D\Phi^a$, where $D$ is the usual supercovariant derivative, and an interaction term of the form $W = t_{abcd}\Phi^a\Phi^b\Phi^c\Phi^d$. The dimensionless symmetric tensor $t_{abcd}$ is restricted by the requirement of $G$ invariance. In terms of component fields we obtain the matter lagrangian

$$\mathcal{L}^0_{\rm m} = \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a + \frac{i}{2}\bar{\psi}^a\gamma^\mu\partial_\mu\psi^a + \frac{1}{2}C^aC^a + t_{abcd}\phi^a\phi^b\left(\frac{1}{3}\phi^cC^d - \frac{1}{2}\bar{\psi}^c\psi^d\right). \tag{3.5}$$

Note that elimination of the auxiliary field $C$ would give a term of the structure $\phi^6$.

The supersymmetry transformations that leave this lagrangian invariant (up to a total derivative) are

$$\delta\phi^a = \bar{\varepsilon}\psi^a = \bar{\psi}^a\varepsilon \tag{3.6}$$

$$\delta\psi^a = -i\gamma^\mu\varepsilon\partial_\mu\phi^a + C^a\varepsilon \tag{3.7}$$

$$\delta C^a = -i\bar{\varepsilon}\gamma^\mu\partial_\mu\psi^a = i\partial_\mu\bar{\psi}^a\gamma^\mu\varepsilon\,. \tag{3.8}$$

[2]For the most part, we follow the conventions of ref. [4]. The metric has signature $+--$ and $\bar{\chi} = \chi^T\gamma^0$. (The transpose here only acts on the spinor components and not on the Lie algebra matrix.) A possible choice of the Dirac matrices in terms of standard Pauli matrices is $\gamma^0 = \sigma_2$, $\gamma^1 = i\sigma_3$, and $\gamma^2 = i\sigma_1$. Note that then $\gamma^{\mu\nu\rho} = -i\epsilon^{\mu\nu\rho}$ and $\gamma^{\mu\nu} = -i\epsilon^{\mu\nu\rho}\gamma_\rho$.



JHEP11(2004)078

This algebra also has off-shell closure giving a translation with the same parameter as in the case of the gauge supermultiplet.

When this system is coupled to the gauge supermultiplet, so as to achieve local $G$ symmetry and global supersymmetry, the supersymmetry transformations of the gauge supermultiplet are unchanged and the supersymmetry transformations of the matter multiplets get a few additional terms (described in the next section) that are required for them to be covariant. Then the commutator of two supersymmetry transformations of the matter fields gives rise to a gauge transformation as well as a translation, with the same parameters as in the case of the gauge supermultiplet discussed earlier.

### 3.3 The gauged theory

We can now put the ingredients together to define the most general gauge-invariant $\mathcal{N}=1$ theory that has classical scale invariance. For this purpose it is convenient to represent the gauge fields by matrices $(A_\mu)^a{}_b$ and $\chi^a{}_b$ in the representation $R$ of the Lie algebra. The total lagrangian is $\mathcal{L} = \mathcal{L}_{\rm CS} + \mathcal{L}_{\rm m}$, where $\mathcal{L}_{\rm CS}$ is given in eq. (3.1) and $\mathcal{L}_{\rm m}$ is eq. (3.5) embellished by couplings to the gauge supermultiplet. The gauged matter lagrangian takes the form

$$\mathcal{L}_{\rm m} = \frac{1}{2}(\nabla_\mu\phi)^a(\nabla^\mu\phi)^a + \frac{i}{2}\bar{\psi}^a\gamma^\mu(\nabla_\mu\psi)^a + \frac{1}{2}C^aC^a + i\phi^a\bar{\chi}^{ab}\psi^b + t_{abcd}\phi^a\phi^b\left[\frac{1}{3}\phi^cC^d - \frac{1}{2}\bar{\psi}^c\psi^d\right], \tag{3.9}$$

where $\nabla_\mu\Phi^a = \partial_\mu\Phi^a + i(A_\mu)^{ab}\Phi^b$.

The supersymmetry transformations of the combined system are

$$\delta A_\mu = i\bar{\varepsilon}\gamma_\mu\chi \tag{3.10}$$

$$\delta\chi = \frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}\,\varepsilon \tag{3.11}$$

$$\delta\phi^a = \bar{\varepsilon}\psi^a \tag{3.12}$$

$$\delta\psi^a = -i\gamma^\mu\varepsilon(\nabla_\mu\phi)^a + C^a\varepsilon \tag{3.13}$$

$$\delta C^a = -i\bar{\varepsilon}\gamma^\mu(\nabla_\mu\psi)^a + i\bar{\varepsilon}\chi^{ab}\phi^b\,. \tag{3.14}$$

The only change from before is the replacement of ordinary derivatives by covariant derivatives and the addition of the second term in $\delta C^a$.

## 4. $\mathcal{N}=2$ models

Let us now try to find models that have $\mathcal{N}=2$ supersymmetry. Since $\mathcal{N}=2$ in three dimensions is closely related to $\mathcal{N}=1$ in four dimensions, and also has a U(1) R symmetry, a complex notation is convenient. For previous related work see [6]–[8].

### 4.1 The gauge multiplet

The Chern-Simons part of the action is constructed out of a vector supermultiplet that can be obtained by dimensional reduction of a four-dimensional $\mathcal{N}=1$ supermultiplet. In four dimensions the multiplet contains a gauge field $A_\mu$, a four-component Majorana spinor $\chi$, and a real scalar $D$. On reduction to three dimensions, the gauge field gives a three-vector gauge field $A_\mu$ and a scalar $\sigma$, corresponding to the component $A_3$ in four dimensions. The spinor can be recast as a two-component Dirac spinor $\chi$, and we still have the scalar $D$. Off-shell there are four bosonic and four fermionic degrees of freedom. In the Chern-Simons theory that we will construct, there are no propagating on-shell degrees of freedom.

Note that the dimension of $A_\mu$ and $\sigma$ is 1, the dimension of $\chi$ is 3/2, and the dimension of $D$ is 2. In terms of an infinitesimal Dirac spinor $\varepsilon$, the supersymmetry transformations for a nonabelian gauge multiplet are the following

$$\delta A_\mu = \frac{i}{2}(\bar{\varepsilon}\gamma_\mu\chi - \bar{\chi}\gamma_\mu\varepsilon) \tag{4.1}$$

$$\delta\sigma = \frac{i}{2}(\bar{\varepsilon}\chi - \bar{\chi}\varepsilon) \tag{4.2}$$

$$\delta D = \frac{1}{2}(\bar{\varepsilon}\gamma^\mu\nabla_\mu\chi + \nabla_\mu\bar{\chi}\gamma^\mu\varepsilon) + \frac{i}{2}(\bar{\varepsilon}[\chi,\sigma] + [\bar{\chi},\sigma]\varepsilon) \tag{4.3}$$

$$\delta\chi = \left(\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu} - iD - \gamma^\mu\nabla_\mu\sigma\right)\varepsilon\,. \tag{4.4}$$

The hermitean conjugate of the last formula is

$$\delta\bar{\chi} = \bar{\varepsilon}\left(-\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu} + iD - \gamma^\mu\nabla_\mu\sigma\right). \tag{4.5}$$

The commutator of two supersymmetry transformations gives a translation by an amount

$$a^\rho = i(\bar{\varepsilon}_1\gamma^\rho\varepsilon_2 - \bar{\varepsilon}_2\gamma^\rho\varepsilon_1) \tag{4.6}$$

and a gauge transformation with parameter

$$\Lambda = -a^\rho A_\rho + i(\bar{\varepsilon}_1\varepsilon_2 - \bar{\varepsilon}_2\varepsilon_1)\sigma. \tag{4.7}$$

We can now construct a supersymmetric Chern-Simons action out of this supermultiplet. The result is

$$\mathcal{L}_{\mathrm{CS}} = \mathrm{tr}\left[\epsilon^{\mu\nu\rho}\left(A_\mu\partial_\nu A_\rho + \frac{2i}{3}A_\mu A_\nu A_\rho\right) - \bar{\chi}\chi + 2D\sigma\right]. \tag{4.8}$$

Note that each of the terms is dimension three.

### 4.2 The matter theory

The notation now is that the index for the matter representation $R$ of the gauge group $G$ is not displayed explicitly, but another index $A$ labelling repetitions of $R$ is displayed. If the matter representation $R$ is complex, let us use the notation $(\Phi^A)^\star = \Phi_A$ to distinguish holomorphic fields and their antiholomorphic conjugates. These can be identified as three-dimensional counterparts of chiral and antichiral superfields in four dimensions. The multiplet contains a complex scalar $\phi^A$ of dimension 1/2, a Dirac two-component spinor $\psi^A$ of dimension 1, and a complex auxiliary scalar $F$ of dimension 3/2. We also have the following R charge assignments: $\phi^A$ has R charge 1/2, $\psi^A$ has R charge $-1/2$, and $F^A$

has R charge $-3/2$. (The conjugates take the negatives of these values, of course.) These conventions correspond to the holomorphic superspace coordinate $\theta$ having R charge 1, and the supersymmetry parameter $\varepsilon$ having R charge $-1$.

We can now write down the supersymmetry transformations for this multiplet. They are

$$\delta\phi^A = \bar{\varepsilon}\psi^A \tag{4.9}$$
$$\delta\psi^A = -i\gamma^\mu\partial_\mu\phi^A\varepsilon + F^A\varepsilon^\star \tag{4.10}$$
$$\delta F^A = -i\bar{\varepsilon}^\star\gamma^\mu\partial_\mu\psi^A\,. \tag{4.11}$$

These formulas are determined, up to coefficients, by dimensional analysis and R symmetry. One can verify that the supersymmetry algebra closes off-shell giving the same translation parameter as for the gauge multiplet.

The $\mathcal{N}=2$ supersymmetric matter lagrangian takes the form

$$\mathcal{L}_{\rm m} = \partial_\mu\phi_A\partial^\mu\phi^A + i\bar{\psi}_A\gamma^\mu\partial_\mu\psi^A + F_AF^A + W_F + W_F^\star\,. \tag{4.12}$$

Here $W_F$ and $W_F^\star$ represent superpotential F-terms, which need to be quartic for scale invariance. As before, they give terms of the form $\phi^2\psi^2$ and $\phi^3F$. The overall normalization of the lagrangian is arbitrary.

### 4.3 The gauged theory

In the gauged theory the supersymmetry transformations of the matter supermultiplet take the form

$$\delta\phi^A = \bar{\varepsilon}\psi^A \tag{4.13}$$
$$\delta\psi^A = (-i\gamma^\mu\nabla_\mu\phi^A - \sigma\phi^A)\varepsilon + F^A\varepsilon^\star \tag{4.14}$$
$$\delta F^A = \bar{\varepsilon}^\star(-i\gamma^\mu\nabla_\mu\psi^A + i\chi\phi^A + \sigma\psi^A)\,. \tag{4.15}$$

For these rules the commutator of two supersymmetry transformations gives a translation and a gauge transformation with the same parameters as for the gauge supermultiplet.

The matter lagrangian that is invariant (up to a total derivative) under these transformations is

$$\begin{aligned}\mathcal{L}_{\rm m} = {} & (\nabla_\mu\phi)_A(\nabla^\mu\phi)^A + i\bar{\psi}_A\gamma^\mu(\nabla_\mu\psi)^A + F_AF^A - \\ & - \phi_A\sigma^2\phi^A + \phi_AD\phi^A - \bar{\psi}_A\sigma\psi^A + i\phi_A\bar{\chi}\psi^A - i\bar{\psi}_A\chi\phi^A + W_F + W_F^\star\,.\end{aligned} \tag{4.16}$$

Combining this with the Chern-Simons terms in eq. (4.8), it is straightforward to eliminate the auxiliary fields $\sigma$, $D$, $\chi$, and $F$. This gives rise to various $\phi^2\psi^2$ and $\phi^6$ terms.

## 5. The $\mathcal{N}=8$ theory?

The free U(1) theory that is the low-energy effective world-sheet theory of an M2-brane in 11 dimensions is well-known [9, 10]. The matter field content consists of scalars $\phi^I$ in the $8_{\rm v}$ representation of Spin(8) and Majorana spinors $\psi^A$ in the $8_{\rm s}$ representation. The eight supersymmetries belong to the $8_c$ representation, and the parameters can be denoted $\varepsilon^{\dot{A}}$. The assignment of these representations is arbitrary, because of triality symmetry. However, the association of the scalars with the vector representation is a natural choice, because they describe excitations of the M2-brane in the eight transverse directions. If one were to add a decoupled U(1) gauge field described by a Chern-Simons action, this would be rather inconsequential, since it has no propagating degrees of freedom and is supersymmetric by itself, as was explained in section 2.

The free matter lagrangian is

$$\mathcal{L}_{\mathrm{m}} = \partial_\mu \phi^I \partial^\mu \phi^I + i\bar{\psi}^A \gamma^\mu \partial_\mu \psi^A \,. \tag{5.1}$$

This is invariant (up to total derivatives) under the supersymmetry transformations

$$\delta\phi^I = \bar{\varepsilon}^{\dot{A}} \Gamma^I_{\dot{A}A} \psi^A \tag{5.2}$$

$$\delta\psi^A = -i\Gamma^I_{A\dot{A}} \gamma^\mu \partial_\mu \phi^I \varepsilon^{\dot{A}} \,, \tag{5.3}$$

where $\Gamma^I_{\dot{A}A}$ and its transpose are invariant tensors that describe the coupling of the three 8s of Spin(8). This is the same structure as in the two-dimensional light-cone gauge world-sheet action for the type IIB superstring in the GS formalism. (The IIA theory uses different representations for left-movers and right-movers.)

The problem now is to find the three-dimensional theory that describes $N$ coincident M2-branes and is the CFT dual of M theory on $AdS_4 \times S^7$ with $N$ units of flux through the seven-sphere. By analogy with the duality between $\mathcal{N} = 4$ SYM theory and type IIB superstring theory on $AdS_5 \times S^5$, it is natural to expect that we need a U($N$) gauge theory (in which the U(1) component decouples) with the matter fields in the adjoint representation. This would be consistent with the idea that they are part of the same supermultiplet as the gauge fields. In that case, it is convenient to represent them by $N \times N$ hermitean matrices. There are reasons for concern, however. One is that the number of degrees of freedom should scale as $N^{3/2}$ for large $N$ [11], whereas the type of construction we are contemplating would appear to give an $N^2$ scaling. Another concern is the parity-conservation issue described in the introduction. We proceed nonetheless with the justification that the existence or nonexistence of an $\mathcal{N} = 8$ U($N$) Chern-Simons theory with the indicated field content is of intrinsic interest irrespective of any possible applications.

A possible approach for constructing the interacting $\mathcal{N} = 8$ theory is to specialize the $\mathcal{N} = 2$ results obtained above to the case where the representation $R$ consists of four complex copies of the adjoint representation. This gives the right field content, and it allows us to make an SU(4) global symmetry (in addition to the U(1) R symmetry) manifest. Once this is achieved, we can try to establish the full $\mathcal{N} = 8$ structure with its Spin(8) R symmetry. This approach is analogous to formulating $\mathcal{N} = 4$ SYM theory in terms of $\mathcal{N} = 1$ superfields. In that formulation only a U(1) $\times$ SU(3) subgroup of the full SU(4) R symmetry is manifest.

In the $\mathcal{N} = 4$ SYM construction there is a superpotential $W = \lambda\, \epsilon_{ABC}\, \mathrm{tr}(\Phi^A \Phi^B \Phi^C)$. When the coefficient $\lambda$ is given the appropriate value, the manifest SU(3) $\times$ U(1) symmetry extends to SU(4), and one obtains $\mathcal{N} = 4$ SYM. In the present problem it seems reasonable to expect an analogous story in which the manifest SU(4) $\times$ U(1) symmetry extends to Spin(8). The superpotential is constructed out of four superfields in the 4 of SU(4). So the analogous superpotential would seem to be $\lambda\,\epsilon_{ABCD}\,\mathrm{tr}(\Phi^A\Phi^B\Phi^C\Phi^D)$. Unfortunately, this vanishes due to the conflicting symmetries of the trace and the epsilon symbol.

The only nonzero possibility appears to be $\epsilon_{ABCD}\,\mathrm{tr}(\Phi^A)\,\mathrm{tr}(\Phi^B\Phi^C\Phi^D)$. Such a formula would imply that the singlet component of the U($N$) fields couples nontrivially. This conflicts with the structure of the rest of the theory, as well as with all expectations. Thus it appears that there cannot be a superpotential. However, without a superpotential contribution, the rest of the theory does not have the desired SO(8) symmetry. This argument constitutes rather strong evidence against the existence of an $\mathcal{N}=8$ theory, at least within the general framework that is being considered here. However, as a check, the problem was also analyzed in terms of component fields with the same conclusion.

## 6. Discussion

We have constructed a class of scale-invariant three-dimensional gauge theories with $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetry, which may be of some interest. For example, gauge theories with three-dimensional conformal invariance could have condensed matter applications [12]. However, our main goal, the construction of scale-invariant gauge theories with $\mathcal{N}=8$ supersymmetry, has not been achieved. There should be a superconformal dual to M theory on $AdS_4 \times S^7$, but since the desired properties are only required at strong coupling, realized as a nontrivial IR fixed point [13]–[15], there need not be a classical lagrangian description.

If $\mathcal{N}=8$ theories of the type that were sought had been shown to exist, there are some interesting questions concerning the AdS/CFT duality that would have arisen. One is the parity issue discussed in the introduction: Chern-Simons theories are parity violating, whereas the super Yang-Mills theory which is supposed to flow to the desired conformal field theory in the IR is not parity violating. Also, M theory is parity conserving. Another concerns the level of the quantum Chern-Simons theory. The $\mathcal{N}=8$ gauge theory would be characterized by two integers: $N$ (the rank of the gauge group) and $k$ (the Chern-Simons level). The level $k$ is expected to be an integer, because the boundary of the Euclideanized M theory geometry is a three-sphere. The gauge coupling would be $g^2 \sim 1/k$. There would be no other continuous parameters. However, two integers is already more than is expected, because the dual geometry is characterized entirely by one integer, the flux through the seven-sphere.

An interesting possibility is that superconformal Chern-Simons theories of the type described here could be dual to $AdS_4 \times K$ compactifications of the massive (Romans) variant of type IIA superstring theory [16].[3] The quantized Romans' mass should correspond to the Chern-Simons level. $AdS_4 \times K$ solutions of massive type IIA supergravity with $\mathcal{N}=1,2,4$ supersymmetry are described in [17, 18]. To pursue this one would also want to study the superconformal symmetry of any proposed dual Chern-Simons theories at the quantum level. (For references on renormalization properties of Chern-Simons theories see [19]–[21].)

[3] I am grateful to Nikita Nekrasov and Greg Moore for this suggestion.

In conclusion, we have constructed large classes of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ classical superconformal Chern-Simons theories, which may be of some interest, but there is no classical $\mathcal{N} = 8$ lagrangian of this type. Moreover, it seems reasonable that there should be no classical lagrangian description of the conformal field theory that is dual to M theory on an $AdS_4 \times S^7$ background.

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